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ENERGY RELEASE RATE CALCULATIONS FOR INTERFACE EDGE CRACKS BASED ON A CONSERVATION INTEGRAL

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Abstract—A certain conservation integral, the so called M-integral, recently exploited by L. B. Freund in the calculation of stress intensity factors for cracks in homogeneous elastic media, is applied to the calculation of energy release rates for interface edge cracks. Specifically, for an edge crack along the interface between two elastic wedges of different opening angles and dissimilar elastic properties, and that is subjected to point loads at the apex, a relation is derived among the length of the crack, the energy release rate of the crack, the applied loads, the wedge angles and the material parameters.

INTRODUCTION

In a recent paper, L. B. Freund[1] demonstrated the usefulness of the M-integral conservation law in the determination of stress intensity factors for 2-dimensional cracks in homogeneous elastic bodies with certain geometric properties. For a detailed discussion of the M-integral and its special features that permit the applications given by Freund, the reader is referred to that paper and the references contained therein. Liberal use will be made in this paper of many of the results derived by Freund.

What is demonstrated here, is that the M-integral can be used to determine the energy release rate for certain interface cracks in much the same way as for cracks in homogeneous bodies.

Three observations are needed for this application. The first, due to Smelzer and Gurtin[2], is that the J-integral on a small arc about an interface crack is equal to the energy release rate, as in the homogeneous case. One difference for cracks in homogeneous bodies is that the J-integral may also be related to the stress intensity factor, while for interface cracks, the J-integral is related to a composite stress intensity factor which has dubious utility (see Smelzer and Gurtin[2]). The second observation involves the nature of the far stress field for bonded dissimilar elastic wedges, and appeals to the analysis presented by Bogy[3] for bonded dissimilar elastic quarter planes. The third observation is that the integrand of the M-integral is continuous across bonded interfaces lying along radial lines of the chosen coordinate system.

STATEMENT AND ANALYSIS OF THE PROBLEM

We present here an extension of the result obtained by Freund for the elastic half-space with an edge crack whose corners are subjected to normal and shear point loads. Specifically, the M-integral conservation law is applied to the 2-dimensional plane strain problem of two infinite isotropic elastic wedges with opening angles ω' and ω'', respectively, and with different elastic properties (exhibited through E', ν' and E'', ν'', where E and ν denote Young's modulus and Poisson's ratio), which are bonded together along one edge except for a crack of length l extending from the apex and whose corners are subjected to a system of normal and shear point loads, given by P', Q', P'' and Q'' (see Fig. 1). An application of the M-integral conservation law yields a relation among the parameters P, Q, E ν, l and the energy release rate of the crack (symbols without primes refer to both materials). It is clear that the

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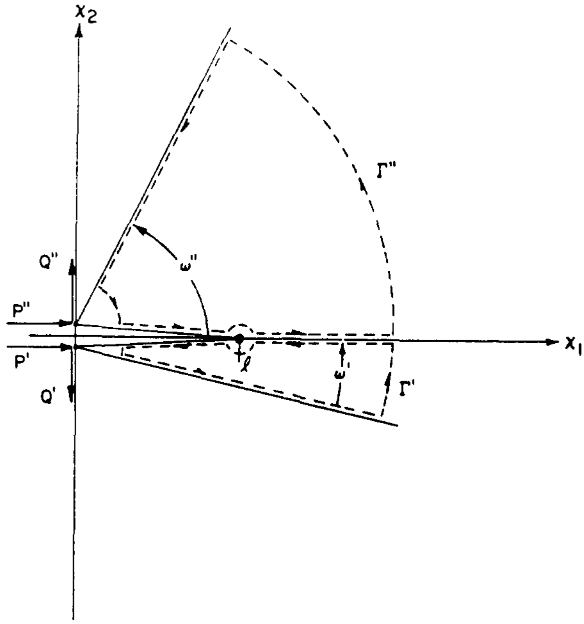

Fig. 1.

complexity of the problem precludes the determination of the energy release rate by first solving the corresponding boundary value problem.

The M-integral is given by

$$M = \int_C (Wn_i x_i - T_k u_{k,i} x_i)\,\mathrm{d}s \tag{1}$$

where W is the elastic energy density, n_i is the unit normal to C (which we take to be directed to the right when C is traversed in a given direction) and T_k is the traction acting on the material to the left of C. The stresses $\sigma_{ij} = \sigma_{ji}$ are related to the elastic energy W and the strains $\epsilon_{ij} = (u_{i,j} + u_{j,i})/2$ by $2W = \sigma_{ij}u_{i,j}$, where u_i denote the displacements. The traction T_k is given by $T_k = \sigma_{ik}n_i$. The elastic body is assumed to be in equlibrium without body forces, i.e. $\sigma_{ij,j} = 0$, and the stress–strain relation is given by $\sigma_{ij} = \partial W/\partial \epsilon_{ij}$. The conservation law for M is that $M = 0$ whenever C is a closed path surrounding a simply connected region in the body.

Consider the integral

$$\begin{aligned} M &= M' + M'' \\ &= \int_\Gamma (Wn_i x_i - T_k u_{k,i} x_i)\,\mathrm{d}s \\ &= \int_{\Gamma'} (W' n'_n x_i - T'_k u'_{k,i} x_i)\,\mathrm{d}s + \int_{\Gamma''} (W'' n''_i x_i - T''_n u''_{k,i} x_i)\,\mathrm{d}s, \end{aligned}$$

where the contours Γ' and Γ'' are as indicated in Fig. 1 and $\Gamma = \Gamma' \cup \Gamma''$. It follows easily from the discussion by Freund, that there is zero contribution to M from the parts of Γ along the crack faces and the outside edges of the wedge. Moreover, the value of M on the vanishingly small arc around the crack tip is $l(\mathrm{d}P/\mathrm{d}l)$, where $P(l)$ is the potential energy of the wedge with a crack of length l. (See Rice[4] and Smelzer and Gurtin[2].) Hence, on this small arc, M is the product of the crack length and the rate of decrease of the energy with respect to crack length.

The contributions to M' and M'' from the small arcs around the corner points of the wedge follow from a general result derived by Freund for an infinite elastic wedge with corner loads. In particular, from the small corner arc in Γ' we obtain the contribution

$$\frac{(1-\nu'^2)}{E'}\left[\frac{F'^2_a}{\omega' + \sin\omega'} + \frac{F'^2_t}{\omega' - \sin\omega'}\right],$$

and from Γ''

$$\frac{(1-\nu''^2)}{E''}\left[\frac{F_a''^2}{\omega''+\sin\omega''}+\frac{F_t''^2}{\omega''-\sin\omega''}\right],$$

where, employing Freund's notation,

$$F_a' = -P'\cos(\omega'/2) - Q'\sin(\omega'/2),$$
$$F_t' = P'\sin(\omega'/2) - Q'\cos(\omega'/2),$$
$$F_a'' = -P''\cos(\omega''/2) - Q''\cos(\omega''/2),$$
$$F_t'' = -P''\sin(\omega''/2) + Q''\cos(\omega''/2).$$

On the bond line, $x_2 = 0$, $x_1 > l$, the contributions to M from Γ' and Γ'' cancel, since, among the stress and displacement components, only σ_{11} is discontinuous across the interface, and on that line $n_i x_i = 0$ (because the interface is radial) and $T_k u_{k,i} x_i = (u_{1,1}\sigma_{21} + u_{2,1}\sigma_{22})x_1$.

It remains to determine the contribution from the large arc for which it suffices to know the far field solution for two wedges bonded together with no crack and with apex loads $P'+P''$ and $Q'-Q''$. From the analysis presented by Bogy[3] for two bonded quarter planes, it is clear that the far field is radial. More precisely, if the elastic fields are represented with respect to polar coordinates (r,θ), then $r\sigma_{\theta\theta}$ and $r\sigma_{r\theta}$ vanish for large r uniformly in θ, whereas $r\sigma_{rr}$ does not. Consequently, on the large arc we may assume $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$ are zero. It is easy to see that in this case, to satisfy the equilibrium equations and the compatability equation, we must take for the stress field

$$\sigma_{rr} = -\frac{A\cos\theta + B\sin\theta}{r}, \qquad \sigma_{r\theta} = \sigma_{\theta\theta} = 0. \tag{2}$$

The four constants A', B', A'' and B'' may be calculated from the four conditions

$$u_\theta'(r,\theta\to 0-) = u_\theta''(r,\theta\to 0+), \tag{3}$$

$$u_r'(r,\theta\to 0-) = u_r''(r,\theta\to 0+), \tag{4}$$

$$\int_{-\omega'}^{0} r\sigma_{rr}'(r,\theta)\cos(\theta)\,\mathrm{d}\theta + \int_0^{\omega''} r\sigma_{rr}''(r,\theta)\cos(\theta)\,\mathrm{d}\theta = -(P'+P''), \tag{5}$$

$$\int_{-\omega'}^{0} r\sigma_{rr}'(r,\theta)\sin(\theta)\,\mathrm{d}\theta + \int_0^{\omega''} r\sigma_{rr}''(r,\theta)\sin(\theta)\,\mathrm{d}\theta = (Q'-Q''). \tag{6}$$

Equations (3) and (4) assert continuity for the displacements on the bond line, while equations (5) and (6) express the equilibrium of tractions on the wedge $-\omega' \le \theta \le \omega''$, $0 \le r \le R$, where R is the radius of the large arc. A simple calculation shows that (5) and (6) reduce to

$$A'(1-\cos 2\omega') - B'(2\omega' - \sin 2\omega') - A''(1-\cos 2\omega'') - B''(2\omega'' - \sin 2\omega'') = 4(Q'-Q'') \tag{7}$$

$$-A'(2\omega' + \sin 2\omega') + B'(1-\cos 2\omega') - A''(2\omega'' + \sin 2\omega'') - B''(1-\cos 2\omega'') = -4(P'+P''). \tag{8}$$

Substitution of (2) into the polar form of the stress-strain law followed by the application of (3) and (4) yields the relations

$$B'/B'' = A'/A'' = (1+\alpha)/(1-\alpha), \tag{9}$$

where α is one of the two Dundurs bi-material parameters given by (see Bogy[3])

$$\alpha = \begin{cases} \dfrac{E'(1-\nu''^2) - E''(1-\nu'^2)}{E'(1-\nu''^2) + E''(1-\nu'^2)} & \text{for plane strain} \\[2ex] \dfrac{E'-E''}{E'+E''} & \text{for generalized plane stress.} \end{cases}$$

It is now an easy matter to solve equations (7), (8) and (9) for A', B', A'' and B''. In particular, we obtain

$$A' = (a(Q'-Q'') + b(P'+P''))(4/d)$$
$$B' = (-a(P'+P'') - c(Q'-Q''))(4/d)$$

where

$$a = (1-\cos 2\omega') - \left(\frac{1-\alpha}{1+\alpha}\right)(1-\cos 2\omega'')$$

$$b = -(2\omega' - \sin 2\omega') - \left(\frac{1-\alpha}{1+\alpha}\right)(2\omega'' - \sin 2\omega'')$$

$$c = -(2\omega' + \sin 2\omega') - \left(\frac{1-\alpha}{1+\alpha}\right)(2\omega'' + \sin 2\omega'')$$

$$d = a^2 - bc.$$

A'' and B'' may now be calculated from (9).

As observed by Freund, for the stress state (2), the integrand of M is

$$Wx_in_i - T_ku_{k,i}x_i = -\frac{1}{2}\frac{(1-\nu^2)}{E}r\sigma_{rr}^2.$$

Consequently, the contribution to M from the large arc is

$$-\frac{1}{2}\frac{(1-\nu'^2)}{E'}\int_{-\omega'}^{0}(A'\cos\theta + B'\sin\theta)^2\,d\theta - \frac{1}{2}\frac{(1-\nu''^2)}{E''}\int_0^{\omega''}(A''\cos\theta + B''\sin\theta)\,d\theta$$

$$= -\frac{1}{2}\frac{(1-\nu')^2}{E'}\left[\int_{-\omega'}^{0}(A'\cos\theta + B'\sin\theta)^2\,d\theta + \left(\frac{1-\alpha}{1+\alpha}\right)\int_0^{\omega''}(A'\cos\theta + B'\sin\theta)\,d\theta\right] \quad (10)$$

$$= -\frac{(1-\nu'^2)}{E}(b(P'+P'')^2 + c(Q'-Q'')^2 + 2a(P'+P'')(Q'-Q''))(2/d). \quad (11)$$

Line (10) follows from (9) and the observation that

$$\frac{(1-\nu''^2)}{E''} = \left(\frac{1+\alpha}{1-\alpha}\right)\frac{(1-\nu'^2)}{E'};$$

whereas, line (11) is derived by simple but tedious algebraic manipulations.

Combining the contributions to M from the large arc, the two small arcs at the apex and the arc around the crack tip and appealing to the conservation law, we obtain

$$l\frac{dP}{dl} = \frac{(1-\nu'^2)}{E'}[-(b(P'+p'')^2 + c(Q'-Q'')^2 + 2a(P'+P'')(Q'-Q''))(2/d)$$
$$+\frac{1}{2}(P'^2(2\omega' - \sin 2\omega') + Q'^2(2\omega' + \sin 2\omega') + 2P'Q'(\cos 2\omega' - 1))/(\omega'^2 - \sin^2\omega') \quad (12)$$
$$+\frac{1}{2}\left(\frac{1+\alpha}{1-\alpha}\right)(P''^2(2\omega'' - \sin 2\omega'') + Q''^2(2\omega'' + \sin 2\omega'') + 2P''Q''(\cos 2\omega'' - 1))/(\omega''^2 - \sin^2\omega'')].$$

An important special case of (12) is that of two bonded dissimilar quarter planes with an edge interface crack. Setting $\omega' = \omega'' = \pi/2$ in (12) yields

$$l\frac{dP}{dl} = \frac{(1-\nu'^2)}{E'}[(P'+P'')^2 + (Q'-Q'')^2 - \frac{4\alpha}{\pi}(P'+P'')(Q'-Q''))(1+\alpha)/((2\alpha)^2 - \pi^2)$$
$$+2\left(\left(P'^2 + \left(\frac{1+\alpha}{1-\alpha}\right)P''^2\right) + \left(Q'^2 + \left(\frac{1+\alpha}{1-\alpha}\right)Q''^2\right) - \frac{4}{\pi}\left(P'Q' + \left(\frac{1+\alpha}{1-\alpha}\right)P''Q''\right)\right)\Big/(\pi^2 - 4)\Big]. \quad (13)$$

It should be noted that if $P' = P'' \equiv P$, $Q' = Q'' \equiv Q$ and $\alpha = 0$, then (13) reduces to the result obtained by Freund for identical quarter planes. Of course, when $\alpha = 0$, the response of the bonded quarter planes is the same as for a homogeneous half-space; as described by Dundurs[5], the two quarter planes are "consonant in tension parallel to the interface."

Another interesting case in (12) is when $\alpha = \pm 1$. Due to symmetry we consider only $\alpha = -1$ corresponding to which (12) becomes

$$l\frac{\mathrm{d}P}{\mathrm{d}l} = \frac{(1-\nu'^2)}{2E'}[P'^2(2\omega' - \sin 2\omega') + Q'^2(2\omega' + \sin 2\omega') + 2P'Q'(\cos 2\omega' - 1))/(\omega'^2 - \sin^2 \omega'). \quad (14)$$

It should not be surprising that in (14) only P' and Q' appear, since $\alpha = -1$ corresponds to $E'' = \infty$.

It should also be noted that for $P' = P'' \equiv P$, $Q' = Q'' \equiv Q$ and $\omega' = \omega'' = \pi/2$, the level curves of $l(\mathrm{d}P/\mathrm{d}l)$ in the (P, Q)-plane are ellipses (straight lines if $\alpha = 0$) centered at $(0, 0)$. From this we may conclude as did Freund, that crack extension may result when the unloading of P and Q occurs along certain paths in the (P, Q)-plane. Moreover, this obviously is the case also in (14) with $\omega' \neq \pi/2$. It is evident in (12), that in general, regardless of the values of ω' and ω'' and α, this phenomenon is to be expected.

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